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A PROOF OF THE THEOREM CONCERNING ARTIFICIAL SINGULARITIES

BY D. R. CURTISS

IN the January number of the *Bulletin of the American Mathematical Society*,* Professor Landau has supplied data lacking in certain fallacious proofs of the well-known theorem which states that if $f(z)$ is single-valued and analytic at all points of the neighborhood of the point c , exclusive of c , and remains finite throughout this neighborhood, it has at most an artificial singularity at c ; i. e., with a suitable definition of $f(c)$ the function $f(z)$ will be analytic at c .†

I wish to point out in this note a simple proof of this theorem which does not assume, as does Professor Landau's paper, the existence of $\lim_{z=c} f(z)$.

The proof makes use of an auxiliary function $\psi(z)$ defined by the equations

$$\psi(z) = (z - c)^2 f(z) \quad (z \neq c), \quad \psi(c) = 0.‡$$

We have

$$\begin{aligned} \psi'(z) &= 2(z - c)f(z) + (z - c)^2 f'(z) \quad (z \neq c), \\ \psi'(c) &= 0. \end{aligned}$$

Thus $\psi(z)$ has a derivative at c as well as throughout its neighborhood. Though Goursat's theorem § makes it unnecessary to prove $\psi'(z)$ continuous

* Vol. 12, pp. 155-156.

† For references and a discussion of various proofs of this theorem see Professor Osgood's paper, "Some points in the elements of the theory of functions," *Bull. Amer. Math. Soc.*, vol. 2 (1896), pp. 296-302.

‡ The auxiliary function considered by Professor Landau and used in the above-mentioned incorrect proofs was the function $\phi(z)$ defined as follows:

$$\phi(z) = (z - c)f(z) \quad (z \neq c), \quad \phi(c) = 0.$$

§ *Trans. Amer. Math. Society*, vol. 1 (1900), pp. 14-16.

at c , we may note that this property is easily deduced from the formula, valid when $|z - c|$ is sufficiently small,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(t) dt}{(t - z)^2},$$

where C is a circumference about z of radius $\frac{1}{2}|z - c|$. If $M > |f(z)|$ throughout the neighborhood of c , exclusive of c , we have

$$|f'(z)| < \frac{2M}{|z - c|};$$

hence

$$\lim_{z=c} \psi'(z) = \psi'(c) = 0.$$

But it can be shown, either by the use of Taylor's series or directly from Cauchy's formula written in the form

$$\begin{aligned} \psi(z) &= \frac{1}{2\pi i} \int \frac{\psi(t) dt}{t - z} \\ &= \frac{1}{2\pi i} \int \frac{\psi(t) dt}{t - c} + \frac{z - c}{2\pi i} \int \frac{\psi(t) dt}{(t - c)^2} + \frac{(z - c)^2}{2\pi i} \int \frac{\psi(t) dt}{(t - z)(t - c)^2} \\ &= \psi(c) + (z - c) \psi'(c) + (z - c)^2 \Psi(z),^* \end{aligned}$$

that on account of the relation $\psi(c) = \psi'(c) = 0$, $\psi(z)$ can be expressed as the product of $(z - c)^2$ and a function $\Psi(z)$ analytic at the point c as well as throughout its neighborhood. We have therefore only to add the definition $f(c) = \Psi(c)$ to make $f(z)$ identical with $\Psi(z)$ throughout the neighborhood of c , inclusive of c .

EVANSTON, ILLINOIS,
JANUARY, 1906.

* The corresponding general form of Taylor's series with the remainder, although capable of wide application, has received little notice in treatises on the theory of functions.